

# Wigner distributions and the joint measurement of incompatible observables

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## Abstract

A theory of joint nonideal measurement of incompatible observables is used in order to assess the relative merits of quantum tomography and certain measurements of generalized observables, with respect to completeness of the obtained information. A method is studied for calculating a Wigner distribution from the joint probability distribution obtained in a joint measurement.

## 1 Quantum tomography

The Wigner distribution  $W(q, p)$  has the well-known Fourier representation

$$W(q, p) = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} d\xi_1 \int_{-\infty}^{\infty} d\xi_2 \tilde{W}(\xi) e^{i\sqrt{2}(p\xi_1 - q\xi_2)},$$

$$\tilde{W}(\xi) = \text{Tr} \hat{\rho} e^{(\xi \hat{a}^\dagger - \xi^* \hat{a})}, \quad \xi = \xi_1 + i\xi_2$$

(carets denote operators). Putting  $\xi = i\eta e^{i\theta}/\sqrt{2}$ , it was observed by Vogel and Risken [1] that  $\tilde{W}(\xi)$  satisfies

$$\tilde{W}(i\eta e^{i\theta}/\sqrt{2}) = \text{Tr} \hat{\rho} e^{i\eta \hat{Q}(\theta)},$$

which is the characteristic function of the rotated quadrature phase operator

$$\hat{Q}(\theta) = \frac{1}{\sqrt{2}}(\hat{a}^\dagger e^{i\theta} + \hat{a} e^{-i\theta}),$$

measured in homodyne optical detection. Since the characteristic function is the Fourier transform of the probability distribution, it was found [1] that the Wigner distribution can be given as the integral

$$W(q, p) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dx \int_0^{2\pi} d\theta \int_0^{\infty} \eta d\eta e^{i\eta x - i\eta(p \sin \theta + q \cos \theta)} w(x, \theta),$$

in which  $w(x, \theta)$  denotes the probability distribution of the rotated quadrature phase observable  $\hat{Q}(\theta)$ . The conclusion that can be drawn from this relation is that the state of the system is completely determined if the probability distributions of all the rotated quadrature phase operators are known. Hence, these operators constitute a so-called quorum [2]. It was remarked by Stenholm [3] that a process of state determination along these lines is similar to the one used in tomography, an obvious disadvantage being its practical intractability because of the necessity of measuring the full probability distribution  $w(x, \theta)$  for *all* angles  $\theta$ .

## 2 Generalized measurements

It was also demonstrated by Stenholm [3] that an improvement in collecting information can be achieved by means of a *simultaneous* measurement procedure of position and momentum proposed by Arthurs and Kelly [4], the joint probability distribution  $P(q, p)$  found in this measurement being expressible in terms of the Wigner distribution  $W(q, p)$  according to

$$P(q, p) = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} dQ \int_{-\infty}^{\infty} dP e^{-\frac{(Q-q)^2}{s^2}} e^{-s^2(P-p)^2} W(Q, P), \quad (1)$$

$s$  an arbitrary real parameter. That the Arthurs-Kelly measurement procedure is indeed a *complete* measurement, determining completely the state  $\hat{\rho}$ , can be seen [5] by inverting (1) by means of deconvolution, thus obtaining

$$W(Q, P) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk' e^{\frac{k^2 s^2}{4} + \frac{k'^2}{4s^2}} \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp e^{ik(q-Q) + ik'(p-P)} P(q, p), \quad (2)$$

the  $k$ -integrals existing if the double Fourier transform of  $P(q, p)$  has asymptotic behaviour  $o(\exp(-\frac{k^2 s^2}{4} - \frac{k'^2}{4s^2}))$ . It is not difficult to prove that this is the case if  $P(q, p)$  is given by (1).

The probability distribution (1) was already found by Husimi [6]. Defining the squeezed states  $\phi_{q,p}^s(x)$  according to

$$\phi_{q,p}^s(x) = (\pi s^2)^{-1/4} e^{-\frac{(x-q)^2}{2s^2} + ix(p-q/2)},$$

the Husimi distribution can be represented as

$$P(q, p) = \text{Tr} \hat{\rho} \frac{1}{2\pi} |\phi_{q,p}^s\rangle \langle \phi_{q,p}^s|. \quad (3)$$

Hence, the parameter  $s$  in (1) is just the squeezing parameter.

The essential point to be noted is, that a measurement yielding the probability distribution (3) is not described by a projection-valued measure generated by the orthogonal spectral resolution of some selfadjoint operator, but by a *positive operator-valued measure* generated by the set of positive operators

$$\hat{M}(q, p) = \frac{1}{2\pi} |\phi_{q,p}^s\rangle \langle \phi_{q,p}^s|, \quad (4)$$

satisfying

$$\hat{M}(q, p) \geq \hat{O}, \quad \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp \hat{M}(q, p) = \hat{I}.$$

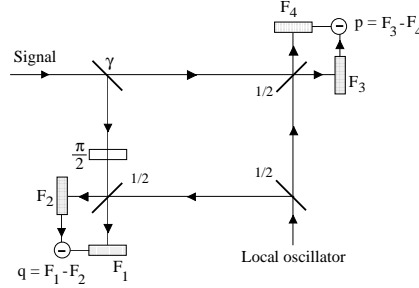


Figure 1: Eight-port optical homodyning as a joint nonideal measurement of position and momentum observables

Measurements described by positive operator-valued measures that are not projection-valued measures are called *generalized* measurements, measuring *generalized* observables. Such measurements have been introduced by Davies [7], Holevo [8] and Ludwig [9], and are studied intensively by now. For instance, it has been demonstrated [10, 11] that the eight-port homodyning technique of detecting monochromatic optical signals, in which the signal is mixed in a Mach-Zehnder interferometer with a sufficiently strong local oscillator field of the same frequency (cf. fig.1), yields (1) for the joint probability of the balanced signals  $q = F_1 - F_2$  and  $p = F_3 - F_4$ , if the transparencies of the mirrors are chosen as indicated in the figure. The parameter  $s$  is determined by the transparency  $\gamma$  according to

$$s^2 = \frac{\gamma}{1 - \gamma}.$$

The technique of optical homodyning is known to induce excess quantum noise in the signal. As a matter of fact, calculating the marginals of  $\hat{M}(q, p)$  we find

$$\hat{M}(q) = \int_{-\infty}^{\infty} dp \hat{M}(q, p) = \int_{-\infty}^{\infty} dq' \frac{1}{\delta_1 \sqrt{\pi}} e^{-\frac{(q-q')^2}{\delta_1^2}} |q'\rangle \langle q'|, \quad \delta_1^2 = s^2,$$

$$\hat{M}(p) = \int_{-\infty}^{\infty} dq \hat{M}(q, p) = \int_{-\infty}^{\infty} dp' \frac{1}{\delta_2 \sqrt{\pi}} e^{-\frac{(p-p')^2}{\delta_2^2}} |p'\rangle \langle p'|, \quad \delta_2^2 = s^{-2}.$$

This result can be interpreted as follows. If  $s = 0$  we have  $\hat{M}(q) = |q\rangle \langle q|$ . Hence, if the partly transparent mirror is replaced by a completely reflecting one ( $\gamma = 0$ ), then the homodyning experiment can be interpreted as an ideal measurement of the quadrature observable  $\hat{Q} = 1/\sqrt{2}(\hat{a}^\dagger + \hat{a})$ . On the other hand, this measurement arrangement does not yield any information on the canonically conjugated observable  $\hat{P} = i/\sqrt{2}(\hat{a}^\dagger - \hat{a})$ . If  $\gamma = 1$ , i.e., the partly transparent mirror is removed completely, we have  $s = \infty$ , and the situation is now the complementary one in which ideal information is obtained on  $\hat{P}$ , whereas all information on  $\hat{Q}$  is wiped out. In the intermediate situations of finite  $s$  we can interpret the experiment as a joint nonideal measurement of  $\hat{Q}$  and

$\hat{P}$ , the Gaussian convolutions describing the excess noise induced in the quadrature observables by inserting the partly transparent mirror. Note that the uncertainties  $\delta_1$  and  $\delta_2$  satisfy the relation

$$\delta_1 \delta_2 = 1,$$

thus exhibiting clearly the complementarity present in the joint nonideal measurement of the two incompatible observables  $\hat{Q}$  and  $\hat{P}$ . It is important to note that the complementarity that is involved here can be seen to have no bearing on the initial preparation of the object, i.e., on the state function  $\hat{\rho}$ , since it is a property of the generalized *observable* (4) alone. This is completely in accordance with the interpretation of complementarity as a mutual disturbance of measurement results in a joint measurement of incompatible observables.

### 3 Joint nonideal measurements of incompatible observables

The notion of a joint nonideal measurement of two incompatible observables was discussed in Martens and de Muynck [12]. Restricting for simplicity to (generalized) observables having discrete spectra, the observables represented by the POVMs  $\{\hat{Q}_m\}$  and  $\{\hat{P}_n\}$  are said to be jointly nonideally measurable if a bivariate POVM  $\{\hat{M}_{mn}\}$  exists, the marginals of which satisfying

$$\sum_n \hat{M}_{mn} = \sum_{m'} \lambda_{mm'} \hat{Q}_{m'}, \quad \lambda_{mm'} \geq 0, \quad \sum_m \lambda_{mm'} = 1, \quad (5)$$

$$\sum_m \hat{M}_{mn} = \sum_{n'} \mu_{nn'} \hat{P}_{n'}, \quad \mu_{nn'} \geq 0, \quad \sum_n \mu_{nn'} = 1. \quad (6)$$

The nonideality matrices  $(\lambda_{mm'})$  and  $(\mu_{nn'})$  determine the nonideality of the measurements of observables  $\{\hat{Q}_m\}$  and  $\{\hat{P}_n\}$ , respectively. Often these matrices are invertible, inverses satisfying

$$\sum_{m'} \lambda_{m'm}^{-1} = 1, \quad \sum_{n'} \mu_{n'n}^{-1} = 1, \quad (7)$$

the matrix elements of the inverse matrices being, however, in general not nonnegative. It is interesting to note that, if the inverses both exist, then it is possible in principle to calculate the probability distributions  $\{Tr \hat{\rho} \hat{Q}_m\}$  and  $\{Tr \hat{\rho} \hat{P}_n\}$  from the measured joint probability distribution  $\{Tr \hat{\rho} \hat{M}_{mn}\}$ , thus obtaining from the joint *nonideal* measurement of  $\{\hat{Q}_m\}$  and  $\{\hat{P}_n\}$  *exact* information on their probability distributions. This, in principle, holds true for the eight-port homodyning case, although in actuality the inversion process may be hampered by incomplete knowledge of the joint probability distribution  $P(q, p)$ .

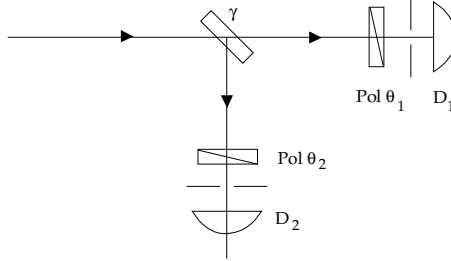


Figure 2: Joint nonideal measurement of two incompatible polarization observables

Defining now an operator-valued measure according to

$$\hat{W}_{m'n'} = \sum_{mn} \lambda_{m'n}^{-1} \mu_{n'n}^{-1} \hat{M}_{mn}, \quad (8)$$

it can easily be verified that (8) satisfies the following relations:

$$\sum_{n'} \hat{W}_{m'n'} = \hat{Q}_{m'}, \quad (9)$$

$$\sum_{m'} \hat{W}_{m'n'} = \hat{P}_{n'}, \quad (10)$$

$$\sum_{m'n'} \hat{W}_{m'n'} = \hat{I}. \quad (11)$$

Because of the possibility that some of the operators  $\hat{W}_{m'n'}$  are not positive, the operator-valued measure is actually a quasi-measure. Since, because of (9) through (11) its expectation values have all the properties of a Wigner distribution, it was called a *Wigner measure*. Applying this procedure to the eight-port homodyning POVM (4) the Wigner measure obtained in this way turns out to have the Wigner distribution (2) as its expectation value.

## 4 Joint nonideal measurement of polarization observables

As a further example we consider the joint nonideal measurement of photon polarization observables. A nonpolarizing beam splitter (transparency  $\gamma$ ), either transmitting a photon toward a polarizer having direction  $\theta_1$  or reflecting it toward a polarizer having direction  $\theta_2$  (cf. fig. 2), can be seen to realize a joint nonideal measurement of the corresponding polarization observables. Denoting the spectral representations of the two observables by  $\{\hat{E}_-^1, \hat{E}_+^1\}$  and  $\{\hat{E}_-^2, \hat{E}_+^2\}$ , respectively, the detection probabilities

of detectors  $D_1$  and  $D_2$  are given by  $\gamma \text{Tr} \rho \hat{E}_m^1$  and  $(1 - \gamma) \text{Tr} \rho \hat{E}_n^2$ , respectively,  $m$  and  $n$  both having the two possible values “yes = +” and “no = -” corresponding to the two possible responses of the detectors. The joint detection probabilities for the two detectors are then easily found as the expectation values of the bivariate positive operator-valued measure generated by the operators  $\hat{M}_{mn}$  defined by

$$(\hat{M}_{mn}) = \begin{pmatrix} \hat{O} & \gamma \hat{E}_+^1 \\ (1 - \gamma) \hat{E}_+^2 & 1 - \gamma \hat{E}_+^1 - (1 - \gamma) \hat{E}_+^2 \end{pmatrix}. \quad (12)$$

Calculating the marginals it is seen that (5) and (6) are satisfied:

$$\begin{pmatrix} \sum_n \hat{M}_{+n} \\ \sum_n \hat{M}_{-n} \end{pmatrix} = \begin{pmatrix} \gamma & 0 \\ 1 - \gamma & 1 \end{pmatrix} \begin{pmatrix} \hat{E}_+^1 \\ \hat{E}_-^1 \end{pmatrix}, \quad (13)$$

$$\begin{pmatrix} \sum_m \hat{M}_{m+} \\ \sum_m \hat{M}_{m-} \end{pmatrix} = \begin{pmatrix} 1 - \gamma & 0 \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} \hat{E}_+^2 \\ \hat{E}_-^2 \end{pmatrix}. \quad (14)$$

From the inverses

$$(\lambda^{-1}) = \begin{pmatrix} \gamma^{-1} & 0 \\ 1 - \gamma^{-1} & 1 \end{pmatrix}, \quad (\mu^{-1}) = \begin{pmatrix} (1 - \gamma)^{-1} & 0 \\ 1 - (1 - \gamma)^{-1} & 1 \end{pmatrix}$$

of the nonideality matrices  $(\lambda)$  and  $(\mu)$  the Wigner measure corresponding to this measurement arrangement can be found according to

$$(\hat{W}_{kl}) = \begin{pmatrix} \hat{O} & \hat{E}_+^1 \\ \hat{E}_+^2 & \hat{E}_-^1 - \hat{E}_+^2 \end{pmatrix}. \quad (15)$$

Note that, contrary to the eight-port homodyning case, the expectation values of the Wigner measure (15) do not determine the state  $\hat{\rho}$  completely. Hence, this measurement is not a complete measurement.

It is not difficult to devise a polarization measurement that is complete. Consider the arrangement of fig. 3 in which three partly transparent mirrors are directing the photon toward one out of four polarizers arranged along four different directions  $\theta_1$  through  $\theta_4$ . The joint probability distribution of this experiment can be found as the expectation values of the operators

$$\begin{aligned} \hat{M}_{+---} &= \gamma_1 \gamma_2 \hat{E}_+^1, \\ \hat{M}_{-+--} &= \gamma_1 (1 - \gamma_2) \hat{E}_+^2, \\ \hat{M}_{--+-} &= (1 - \gamma_1) \gamma_3 \hat{E}_+^3, \\ \hat{M}_{----+} &= (1 - \gamma_1) (1 - \gamma_3) \hat{E}_+^4, \\ \hat{M}_{-----} &= 1 - (\hat{M}_{+---} + \hat{M}_{-+--} + \hat{M}_{--+-} + \hat{M}_{----+}), \end{aligned}$$

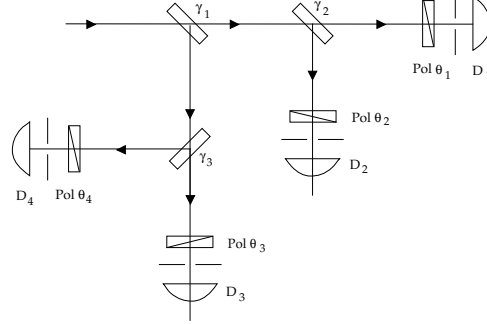


Figure 3: Joint nonideal measurement of four incompatible polarization observables

generating the positive operator-valued measure describing the experiment (all  $\hat{M}_{ijk\ell}$  having more than one + vanishing). We can order these operators in a way naturally generalizing (12) so as to obtain a *bivariate* POVM:

$$(\hat{M}_{mn}) = \begin{pmatrix} \hat{O} & \gamma_1 \gamma_2 \hat{E}_+^1 & \hat{O} & \hat{O} \\ \gamma_1(1-\gamma_2)\hat{E}_+^2 & \gamma_1[\hat{I} - \gamma_2\hat{E}_+^1 - (1-\gamma_2)\hat{E}_+^2] & \hat{O} & \hat{O} \\ \hat{O} & \hat{O} & \hat{O} & (1-\gamma_1)\gamma_3\hat{E}_+^3 \\ \hat{O} & \hat{O} & (1-\gamma_1)(1-\gamma_3)\hat{E}_+^4 & (1-\gamma_1)[\hat{I} - \gamma_3\hat{E}_+^3 - (1-\gamma_3)\hat{E}_+^4] \end{pmatrix}.$$

Now this measurement can be seen to represent a joint nonideal measurement of the observables represented by the POVMs

$$\{\hat{Q}_m\} = \{\gamma_1\hat{E}_+^1, \gamma_1\hat{E}_-^1, (1-\gamma_1)\hat{E}_+^3, (1-\gamma_1)\hat{E}_-^3\} \quad (16)$$

and

$$\{\hat{P}_n\} = \{\gamma_1\hat{E}_+^2, \gamma_1\hat{E}_-^2, (1-\gamma_1)\hat{E}_+^4, (1-\gamma_1)\hat{E}_-^4\}, \quad (17)$$

the two marginals  $\{\sum_n \hat{M}_{mn}\}$  and  $\{\sum_m \hat{M}_{mn}\}$  being expressible according to (5) and (6) in the POVMs (16) and (17) with nonideality matrices

$$(\lambda) = \begin{pmatrix} \gamma_2 & 0 & 0 & 0 \\ 1-\gamma_2 & 1 & 0 & 0 \\ 0 & 0 & \gamma_3 & 0 \\ 0 & 0 & 1-\gamma_3 & 1 \end{pmatrix}, \quad (\mu) = \begin{pmatrix} 1-\gamma_2 & 0 & 0 & 0 \\ \gamma_2 & 1 & 0 & 0 \\ 0 & 0 & 1-\gamma_3 & 0 \\ 0 & 0 & \gamma_3 & 1 \end{pmatrix}.$$

Note that the POVMs (16) and (17) are maximal POVMs in the sense that neither of them does describe a nonideal measurement of any inequivalent generalized observable (cf. [12]).

The matrices  $(\lambda)$  and  $(\mu)$  are invertible. Calculating the Wigner measure (8) for this measurement we find

$$(\hat{W}_{mn}) = \begin{pmatrix} \hat{O} & \gamma_1 \hat{E}_+^1 & \hat{O} & \hat{O} \\ \gamma_1 \hat{E}_+^2 & \gamma_1 (\hat{E}_-^1 - \hat{E}_+^2) & \hat{O} & \hat{O} \\ \hat{O} & \hat{O} & \hat{O} & (1 - \gamma_1) \hat{E}_+^3 \\ \hat{O} & \hat{O} & (1 - \gamma_1) \hat{E}_+^4 & (1 - \gamma_1) (\hat{E}_-^3 - \hat{E}_+^4) \end{pmatrix}.$$

It is easily verified that this Wigner measure has the POVMs (16) and (17) as its marginals. It is interesting to note that, contrary to the joint nonideal measurement of two conventional observables like those represented by the projection-valued measures  $\{\hat{E}_-^1, \hat{E}_+^1\}$  and  $\{\hat{E}_-^2, \hat{E}_+^2\}$ , the joint nonideal measurement of the two *generalized* observables (16) and (17) is a complete measurement.

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